GAUSS TEST

— Let $\sum u_n$ be a +ve term series and let there exists 2 +ve numbers ρ and δ and a bounded sequence $\langle a_n \rangle$ such that

$$\frac{u_n}{u_{n+1}} = 1 + \frac{\rho}{n} + \frac{a_n}{n^{1+\delta}}$$

then the series $\sum u_n$
converges if $\rho > 1$
and diverges if $\rho \leq 1$

Proof:-

Given that

$$\frac{u_n}{u_{n+1}} = 1 + \frac{\rho}{n} + \frac{a_n}{n^{1+\delta}}$$

then

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = \rho + \frac{a_n}{n^{\delta}}$$

$$\therefore \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty}\left(\rho + a_n \cdot \frac{1}{n^{\delta}}\right)$$

$$= \rho \quad \text{(as } \delta \text{ is +ve}$$

So, by Raabe's Test the series converges if $\rho > 1$ and diverges if $\rho < 1$

Now, consider $\rho = 1$

In this case

$$\frac{u_n}{u_{n+1}} = 1 + \frac{1}{n} + \frac{a_n}{n^{1+\delta}} \quad \text{--- (1)}$$

Consider the divergent series

$$\Sigma v_n = \Sigma \frac{1}{n \log n}$$

$$\frac{u_n}{u_{n+1}} - \frac{v_n}{v_{n+1}} = 1 + \frac{2}{n} + \frac{a_n}{n^{1+\delta}} - \frac{(n+1)\log(n+1)}{n \log n}.$$

$$= \frac{a_n}{n^{1+\delta}} - \frac{n+1}{n}\left[\frac{\log(n+1)}{\log n} - 1\right]$$

$$= \frac{1}{n^{1+\delta}}\left[a^n - (n+1)\log\left(1+\frac{1}{n}\right) \cdot \frac{n^{\delta}}{\log n}\right] \quad \text{--- (2)}$$

But,

$$\lim_{n\to\infty} (n+1)\log\left(1+\frac{1}{n}\right)$$

$$= \lim_{n\to\infty} n\log\left(1+\frac{1}{n}\right) + \log\left(1+\frac{1}{n}\right).$$

$$= \lim_{n\to\infty} \left[\log\left(1+\frac{1}{n}\right)^n + \log\left(1+\frac{1}{n}\right)\right]$$

$$= 1.$$

Also, $\lim_{n\to\infty} \frac{n^\delta}{\log n} = \infty$

L-Hospitals Rule
$\lim_{n\to\infty} \frac{\delta n^{\delta-1}}{1/n}$
$= \lim_{n\to\infty} \delta n^\delta = \infty$

Thus, we see that for sufficiently large value of n

$$\left\{a_n - (n+1)\log\left(1+\frac{1}{n}\right)\frac{n^\delta}{\log n}\right\} < 0 \quad \text{——— } (3)$$

From (2) and (3).

$$\frac{u_n}{u_{n+1}} - \frac{v_n}{v_{n+1}} < 0 \quad \forall\ n \geq 0$$

ie. $\frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}}$

$\therefore$ By comparision Test Σu_n is divergent.

8). Discuss the Convergency of the series
$x^2(\log 2)^p + x^3(\log 3)^p + \cdots \cdots + x^n(\log n)^p + \cdots\cdots$

Solution:

$$u_n = x^{n+1}\{\log(n+1)\}^p$$

$$u_{n+1} = x^{n+2}\{\log(n+2)\}^p$$

$$\frac{u_n}{u_{n+1}} = \frac{1}{x}\left\{\frac{\log(n+1)}{\log(n+2)}\right\}^p$$

$$= \frac{1}{x}\left\{\frac{\log n(1+\frac{1}{n})}{\log n(1+\frac{2}{n})}\right\}^p$$

$$= \frac{1}{x}\cdot\left\{\frac{\log n + \log(1+\frac{1}{n})}{\log n + \log(1+\frac{2}{n})}\right\}^p$$

$$= \frac{1}{x}\left\{\frac{\log n + \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{\ } \cdots\cdots}{\log n + \frac{2}{n} - \frac{4}{2n^2} + \cdots\cdots}\right\}^p.$$

$$= \frac{1}{x}\left\{\frac{1 + \frac{1}{n\log n} - \cdots\cdots}{1 + \frac{2}{n\log n} - \cdots}\right\}^p$$

$$= \frac{1}{x}\left(1+\frac{1}{n\log n}+\cdots\right)^{p}\left(1+\frac{2}{n\log n}+\cdots\right)^{-p}$$

$$= \frac{1}{x}\left(1+\frac{p}{n\log n}+\cdots\right)\left(1-\frac{2p}{n\log n}+\cdots\right)$$

$$= \frac{1}{x}\left(1+\frac{p}{n\log n}-\frac{2p}{n\log n}+\cdots\right)$$

$$= \frac{1}{x}\left(1-\frac{p}{n\log n}\right)$$

$$\therefore \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \frac{1}{x}$$

So, by D'Alembert's Ratio Test the given series is convergent if $\frac{1}{x} > 1$

i.e. $x < 1$

and divergent if $\frac{1}{x} < 1$

i.e $x > 1$

when $x = 1$, D'Alembert's ratio test fails.

In this case

$$\frac{u_n}{u_{n+1}} = 1 - \frac{p}{n\log n}$$

$$\therefore \lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}} - 1\right] = \lim_{n\to\infty} n\left[\frac{-p}{n\log n}\right]$$

$$= \lim_{n\to\infty} \frac{-p}{\log n}$$

$$= 0 < 1$$

So, By Raabe's Test, series is divergent for $x = 1$.

Hence, the given series is convergent when $x < 1$
and divergent when $x \geq 1$.